

A note on Todorov surfaces

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Abstract

Let S be a *Todorov surface*, *i.e.*, a minimal smooth surface of general type with $q = 0$ and $p_g = 1$ having an involution i such that S/i is birational to a $K3$ surface and such that the bicanonical map of S is composed with i .

The main result of this paper is that, if P is the minimal smooth model of S/i , then P is the minimal desingularization of a double cover of \mathbb{P}^2 ramified over two cubics. Furthermore it is also shown that, given a Todorov surface S , it is possible to construct Todorov surfaces S_j with $K^2 = 1, \dots, K_S^2 - 1$ and such that P is also the smooth minimal model of S_j/i_j , where i_j is the involution of S_j . Some examples are also given, namely an example different from the examples presented by Todorov in [To2].

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1 Introduction

An *involution* of a surface S is an automorphism of S of order 2. We say that a map is *composed with an involution* i of S if it factors through the double cover $S \rightarrow S/i$. Involutions appear in many contexts in the study of algebraic surfaces. For instance in most cases the bicanonical map of a surface of general type is non-birational only if it is composed with an involution.

Assume that S is a smooth minimal surface of general type with $q = 0$ and $p_g \neq 0$ having bicanonical map ϕ_2 composed with an involution i of S such that S/i is non-ruled. Then, according to [Xi, Theorem 3], $p_g(S) = 1$, $K_S^2 \leq 8$ and S/i is birational to a $K3$ surface (Theorem 3 of [Xi] contains the assumption $\deg(\phi_2) = 2$, but the result is still valid assuming only that ϕ_2 is composed with an involution).

Todorov ([To2]) was the first to give examples of such surfaces. His construction is as follows. Consider a Kummer surface Q in \mathbb{P}^3 , *i.e.*, a quartic having as only singularities 16 nodes a_i . The double cover of Q ramified over the intersection of Q with a general quadric and over the 16 nodes of Q is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8$. Then, choose a_1, \dots, a_6 in general position and let G be the intersection of Q with a general quadric through j of the nodes a_1, \dots, a_6 . The double cover of Q ramified over $Q \cap G$ and over the remaining $16 - j$ nodes of Q is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8 - j$.

Imposing the passage of the branch curve by a 7-th node, one can obtain a surface with $K^2 = p_g = 1$ and $q = 0$. This is the so-called *Kunev surface*. Todorov ([To1]) has shown that the Kunev surface is a bidouble cover of \mathbb{P}^2 ramified over two cubics and a line.

I refer to [Mo] for an explicit description of the moduli spaces of Todorov surfaces.

We call *Todorov surfaces* smooth surfaces S of general type with $p_g = 1$ and $q = 0$ having bicanonical map composed with an involution i of S such that S/i is birational to a $K3$ surface.

In this paper we prove the following:

Theorem 1 *Let S be a Todorov surface with involution i and P be the smooth minimal model of S/i . Then:*

- a) *there exists a generically finite degree 2 morphism $P \rightarrow \mathbb{P}^2$ ramified over two cubics;*
- b) *for each $j \in \{1, \dots, K_S^2 - 1\}$, there is a Todorov surface S_j , with involution i_j , such that $K_{S_j}^2 = j$ and P is the smooth minimal model of S_j/i_j .*

The idea of the proof is the following. First we verify that the evenness of the branch locus $B' + \sum A_i \subset P$ implies that each nodal curve A_j can only be contained in a Dynkin graph G of type A_{2n+1} or D_n . Then we use a Saint-Donat result to show that A_j can be chosen such that the linear system $|B' - G|$ is free. This implies b). Finally we conclude that there is a free linear system $|B'_0|$ with $B'_0{}^2 = 2$, which gives a).

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface S , the *canonical class* is denoted by K , the *geometric genus* by $p_g := h^0(S, \mathcal{O}_S(K))$, the *irregularity* by $q := h^1(S, \mathcal{O}_S(K))$ and the *Euler characteristic* by $\chi = \chi(\mathcal{O}_S) = 1 + p_g - q$.

A (-2) -curve or *nodal curve* on a surface is a curve isomorphic to \mathbb{P}^1 such that $C^2 = -2$. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up.

The rest of the notation is standard in algebraic geometry.

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2 Preliminaries

The next result follows from [St, (4.1), Theorem 5.2, Propositions 5.6 and 5.7].

Theorem 2 ([St]) *Let $|D|$ be a complete linear system on a smooth $K3$ surface F , without fixed components and such that $D^2 \geq 4$. Denote by φ_D the map given by $|D|$. If φ_D is non-birational and the surface $\varphi_D(F)$ is singular then there exists an elliptic pencil $|E|$ such that $ED = 2$ and one of these cases occur:*

- (i) $D = O_F(4E + 2\Gamma)$ where Γ is a smooth rational irreducible curve such that $\Gamma E = 1$. In this case $\varphi_D(F)$ is a cone over a rational normal twisted quartic in \mathbb{P}^4 ;
- (ii) $D = O_F(3E + 2\Gamma_0 + \Gamma_1)$, where Γ_0 and Γ_1 are smooth rational irreducible curves such that $\Gamma_0 E = 1$, $\Gamma_1 E = 0$ and $\Gamma_0 \Gamma_1 = 1$. In this case $\varphi_D(F)$ is a cone over a rational normal twisted cubic in \mathbb{P}^3 ;
- (iii) a) $D = O_F(2E + \Gamma_0 + \Gamma_1)$, where Γ_0 and Γ_1 are smooth rational irreducible curves such that $\Gamma_0 E = \Gamma_1 E = 1$ and $\Gamma_0 \Gamma_1 = 0$;
b) $D = O_F(2E + \Delta)$, with $\Delta = 2\Gamma_0 + \dots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ ($N \geq 0$), where the curves Γ_i are irreducible rational curves as in Figure 1.

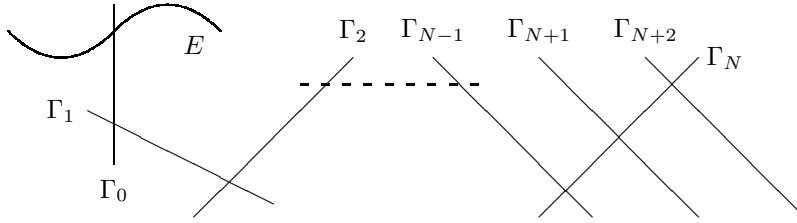


Figure 1: Configuration (iii) b)

In both cases $\varphi_D(F)$ is a quadric cone in \mathbb{P}^3 .

Moreover in all the cases above the pencil $|E|$ corresponds under the map φ_D to the system of generatrices of $\varphi_D(F)$.

3 Proof of Theorem 1

We say that a curve D is *nef* and *big* if $DC \geq 0$ for every curve C and $D^2 > 0$. In order to prove Theorem 1, we show the following:

Proposition 3 *Let P be a smooth K3 surface with a reduced curve B satisfying:*

- (i) $B = B' + \sum_1^t A_i$, $t \in \{9, \dots, 16\}$, where B' is a nef and big curve with at most negligible singularities, the curves A_i are disjoint (-2) -curves also disjoint from B' and $B \equiv 2L$, $L^2 = -4$, for some $L \in \text{Pic}(P)$.

Then:

- a) Let $\pi : V \rightarrow P$ be a double cover with branch locus B and S be the smooth minimal model of V . Then $q(S) = 0$, $p_g(S) = 1$, $K_S^2 = t - 8$ and the bicanonical map of S is composed with the involution i of S induced by π ;
- b) If $t \geq 10$, then P contains a smooth curve B'_0 and (-2) -curves A'_1, \dots, A'_{t-1} such that $B_0'^2 = B'^2 - 2$ and $B_0 := B'_0 + \sum_1^{t-1} A'_i$ also satisfies condition (i).

Proof:

a) Let $L \equiv \frac{1}{2}B$ be the line bundle which determines π . From the double cover formulas (see *e.g.* [BPV]) and the Riemann Roch theorem,

$$q(S) = h^1(P, \mathcal{O}_P(L)),$$

$$p_g(S) = 1 + h^0(P, \mathcal{O}_P(L)),$$

$$h^0(P, \mathcal{O}_P(L)) + h^0(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(L)).$$

Since $2L - \sum A_i$ is nef and big, the Kawamata-Viehweg's vanishing Theorem (see *e.g.* [EV, Corollary 5.12, c)]) implies $h^1(P, \mathcal{O}_P(-L)) = 0$. Hence

$$h^1(P, \mathcal{O}_P(L)) = h^1(P, \mathcal{O}_P(K_P - L)) = h^1(P, \mathcal{O}_P(-L)) = 0$$

and then $q(S) = 0$ and $p_g(S) = 1$. As

$$h^0(P, \mathcal{O}_P(2K_P + L)) = h^0(P, \mathcal{O}_P(L)) = 0,$$

the bicanonical map of S is composed with i (see [CM, Proposition 6.1]).

The (-2) -curves A_1, \dots, A_t give rise to (-1) -curves in V , therefore

$$K_S^2 = K_V^2 + t = 2(K_P + L)^2 + t = 2L^2 + t = t - 8.$$

b) Denote by $\xi \subset P$ the set of irreducible curves which do not intersect B' and denote by ξ_i , $i \geq 1$, the connected components of ξ . Since $B'^2 \geq 2$, the Hodge-index Theorem implies that the intersection matrix of the components of ξ_i is negative definite. Therefore, following [BPV, Lemma I.2.12], the ξ_i 's have one of the five configurations: the support of A_n , D_n , E_6 , E_7 or E_8 (see *e.g.* [BPV, III.3] for the description of these graphs).

Claim 1: *Each nodal curve A_i can only be contained in a graph of type A_{2n+1} or D_n .*

Proof: Suppose that there exists an A_i which is contained in a graph of type E_6 . Denote the components of E_6 as in Figure 2.

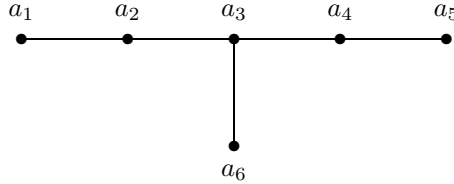


Figure 2: E_6

If $A_i = a_3$ or $A_i = a_6$, then $a_6B = a_6a_3 = 1$ or $a_3B = 1$, contradicting $B \equiv 2L$. If $A_i = a_1$ or $A_i = a_2$, then $a_2B = 1$ or $a_1B = 1$, the same contradiction. By the same reason, $A_i \neq a_4$ and $A_i \neq a_5$.

Analogously one can verify that each A_i can not be in a graph of type A_{2n} , E_7 or E_8 . \diamond

The possible configurations for the curves A_i in the graphs are shown in Figure 3. Fix one of the curves A_i and denote by G the graph containing it.

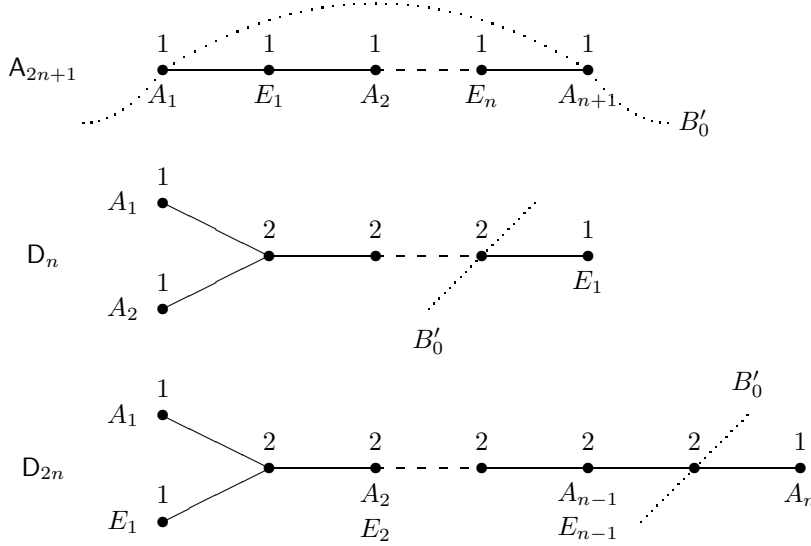


Figure 3: The numbers represent the multiplicity and the dotted curve represent a general element B'_0 in $|B' - G|$.

Claim 2: We can choose A_i such that the linear system $|B' - G|$ has no fixed components (and thus no base points, from [St, Theorem 3.1]).

Proof : Denote by $\varphi_{|B'|}$ the map given by the linear system $|B'|$. We know that $\varphi_{|B'|}$ is birational or it is of degree 2 (see [St, Section 4]). If $\varphi_{|B'|}$ is birational or the point $\varphi_{|B'|}(G)$ is a smooth point of $\varphi_{|B'|}(P)$, the result is clear, since $|B' - G|$ is the pullback of the linear system of the hyperplanes containing $\varphi_{|B'|}(G)$ and $\varphi_{|B'|}^*(\varphi_{|B'|}(G)) = G$ (see [BPV, Theorems III 7.1 and 7.3]).

Suppose now that $\varphi_{|B'|}$ is non-birational and that $\varphi_{|B'|}(G)$ is a singular point of $\varphi_{|B'|}(P)$. Then B' is linearly equivalent to a curve with one of the configurations described in Theorem 2. Except for the last configuration, G contains at most two (-2) -curves. But $t \geq 9$, thus in these cases there exists other graph G' containing a curve A_j such that $\varphi_{|B'|}(G')$ is a non-singular point of $\varphi_{|B'|}(P)$ (notice that Theorem 2 implies that $\varphi_{|B'|}(P)$ contains only one singular point).

So we can suppose that B' is equivalent to a curve with a configuration as in Theorem 2, (iii), b). None of the curves $\Gamma_0, \dots, \Gamma_N$ can be one of the curves A_j . For this note that: if $\Gamma_0 = A_j$, then $EB = E(B' + \sum A_i) = 2 + E\Gamma_0 = 3 \not\equiv 0 \pmod{2}$; if $\Gamma_1 = A_j$, then $\Gamma_0 B = \Gamma_0 \Gamma_1 = 1 \not\equiv 0 \pmod{2}$; etc. Again this configuration can contain at most two curves A_j , the components $\Gamma_{N+1}, \Gamma_{N+2}$. \diamond

Let B'_0 be a smooth curve in $|B' - G|$. If G is an A_{2n+1} graph, then, using the notation of Figure 3,

$$\begin{aligned} \left(B'_0 + \sum_1^n E_i \right) + \sum_{n+2}^t A_i &\equiv \left(B' - \sum_1^{n+1} A_i \right) + \sum_{n+2}^t A_i \equiv \\ &\equiv B' + \sum_1^t A_i - 2 \sum_1^{n+1} A_i \equiv 0 \pmod{2}. \end{aligned}$$

Therefore the curve

$$B_0 := B'_0 + \sum_1^n E_i + \sum_{n+2}^t A_i$$

satisfies condition (i).

The case where G is a D_m graph is analogous. \square

Proof of Theorem 1 : Let $V \rightarrow S$ be the blow-up at the isolated fixed points of the involution i and W be the minimal resolution of S/i . We have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & S \\ \pi \downarrow & & \downarrow \\ W & \longrightarrow & S/i. \end{array}$$

The branch locus of π is a smooth curve $B = B' + \sum_1^t A_i$, where the curves A_i are (-2) -curves which contract to the nodes of S/i . Let P be the minimal model of W and $\overline{B} \subset P$ be the projection of B . Let $L \equiv \frac{1}{2}B$ be the line bundle which determines π .

First we verify that \overline{B} satisfies condition (i) of Proposition 3: from [CM, Proposition 6.1], $\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L)$, hence $K_W(K_W + L) = 0$, which implies that \overline{B} has at most negligible singularities; now from [Mo, Theorem 5.2] we get $K_S^2 = \frac{1}{2}\overline{B'}^2$ and $1 = p_g(S) = \frac{1}{4}(K_S^2 - t) + 3$, thus $t = K_S^2 + 8$ and $\overline{B}^2 = \overline{B'}^2 - 2t = 2K_S^2 - 2t = -16$, which gives $(\overline{B}/2)^2 = -4$ and $\overline{B'}^2 \geq 2$; finally $\overline{B'}$ is nef because, on a $K3$ surface, an irreducible curve with negative self intersection must be a (-2) -curve.

Now using Proposition 3, b) and a) we obtain statement b). In particular we get also that P contains a curve B'_0 and (-2) -curves A'_i , $i = 1, \dots, 9$, such that $B_0 := B'_0 + \sum_1^9 A'_i$ is smooth and divisible by 2 in the Picard group. Moreover, the complete linear system $|B'_0|$ has no fixed component nor base points and $B_0'^2 = 2$. Therefore, from [St], $|B'_0|$ defines a generically finite degree 2 morphism

$$\varphi := \varphi_{|B'_0|} : P \rightarrow \mathbb{P}^2.$$

Since $g(B'_0) = 2$, this map is ramified over a sextic curve β . The singularities of β are negligible because P is a $K3$ surface.

We claim that β is the union of two cubics. Let $p_i \in \beta$ be the singular point corresponding to A'_i , $i = 1, \dots, 9$. Notice that the p_i 's are possibly infinitely near.

Let $C \subset \mathbb{P}^2$ be a cubic curve passing through p_i , $i = 1, \dots, 9$. As $C + \varphi_*(B'_0)$ is a plane quartic, we have

$$\left(\varphi^*(C) - \sum_1^9 A'_i \right) + B'_0 + \sum_1^9 A'_i \equiv \varphi^*(C + \varphi_*(B'_0)) \equiv 0 \pmod{2},$$

hence also $\varphi^*(C) - \sum_1^9 A'_i \equiv 0 \pmod{2}$, *i.e.* there exists a divisor J such that

$$2J \equiv \varphi^*(C) - \sum_1^9 A'_i.$$

Since P is a K_3 surface, the Riemann Roch theorem implies that J is effective. From $JA'_i = 1$, $i = 1, \dots, 9$, we obtain that the plane curve $\varphi_*(J)$ passes with multiplicity 1 through the nine singular points p_i of β . This immediately implies that $\varphi_*(J)$ is not a line nor a conic, because β is a reduced sextic. Therefore $\varphi_*(J)$ is a reduced cubic. So $\varphi_*(J) \equiv C$ and then

$$\varphi^*(\varphi_*(J)) \equiv 2J + \sum_1^9 A'_i.$$

This implies that $\varphi_*(J)$ is contained in the branch locus β , which finishes the proof of a). \square

4 Examples

Todorov gave examples of surfaces S with bicanonical image $\phi_2(S)$ birational to a Kummer surface having only ordinary double points as singularities. The next sections contain an example with $\phi_2(S)$ non-birational to a Kummer surface and an example with $\phi_2(S)$ having an A_{17} double point.

4.1 S/i non-birational to a Kummer surface

Here we construct smooth surfaces S of general type with $K^2 = 2, 3$, $p_g = 1$ and $q = 0$ having bicanonical map of degree 2 onto a $K3$ surface which is not birational to a Kummer surface.

It is known since [Hu] that there exist special sets of 6 nodes, called Weber hexads, in the Kummer surface $Q \in \mathbb{P}^3$ such that the surface which is the blow-up of Q at these nodes can be embedded in \mathbb{P}^3 as a quartic with 10 nodes. This quartic is the Hessian of a smooth cubic surface.

The space of all smooth cubic surfaces has dimension 4 while the space of Kummer surfaces has dimension 3. Thus it is natural to ask if there exist Hessian "non-Kummer" surfaces, *i.e.* which are not the embedding of a Kummer surface blown-up at 6 points. This is studied in [Ro], where the existence of "non-Kummer" quartic Hessians H in \mathbb{P}^3 is shown. These are surfaces with 10 nodes a_i such that the projection from one node a_1 to \mathbb{P}^2 is a generically $2:1$ cover of \mathbb{P}^2 with branch locus $\alpha_1 + \alpha_2$ satisfying: α_1, α_2 are smooth cubics

tangent to a nondegenerate conic C at 3 distinct points. We use this in the following construction.

Let α_1, α_2 and C be as above. Take the morphism $\pi : W \rightarrow \mathbb{P}^2$ given by the canonical resolution of the double cover of \mathbb{P}^2 with branch locus $\alpha_1 + \alpha_2$. The strict transform of C gives rise to the union of two disjoint (-2) -curves $A_1, A_2 \subset W$ (one of these correspond to the node a_1 from which we have projected).

Let $T \in \mathbb{P}^2$ be a general line. Let $A_3, \dots, A_{11} \subset W$ be the disjoint (-2) -curves contained in $\pi^*(\alpha_1 + \alpha_2)$. We have $\pi^*(T + \alpha_1) \equiv 0 \pmod{2}$, hence, since α_1 is in the branch locus, also

$$\pi^*(T) + \sum_3^{11} A_i \equiv 0 \pmod{2}.$$

The linear systems $|\pi^*(T) + A_2|$ and $|\pi^*(T) + A_1 + A_2|$ have no fixed components nor base points (see [St, (2.7.3) and Corollary 3.2]). The surface S is the minimal model of the double cover of W ramified over a general element in

$$|\pi^*(T) + A_2| + \sum_2^{11} A_i \quad \text{or} \quad |\pi^*(T) + A_1 + A_2| + \sum_1^{11} A_i.$$

4.2 $\phi_2(S)$ with A_{17} and A_1 singularities

This section contains a brief description of a construction of a surface S of general type having bicanonical image $\phi_2(S) \subset \mathbb{P}^3$ a quartic $K3$ surface with A_{17} and A_1 singularities. I omit the details, which were verified using the *Computational Algebra System Magma*.

Let C_1 be a nodal cubic, p an inflection point of C_1 and T the tangent line to C_1 at p . The pencil generated by C_1 and $3T$ contains another nodal cubic C_2 , smooth at p . The curves C_1 and C_2 intersect at p with multiplicity 9.

Let $\rho : X \rightarrow \mathbb{P}^2$ be the resolution of $C_1 + C_2$ and $\pi : W \rightarrow X$ be the double cover with branch locus the strict transform of $C_1 + C_2$. Denote by \bar{l} the line containing the nodes of C_1 and C_2 and by $l \subset W$ the pullback of the strict transform of \bar{l} . The map given by $|(\rho \circ \pi)^*(\bar{l}) + l|$ is birational onto a quartic Q in \mathbb{P}^3 with an A_1 and A_{17} singularities (notice that l is a (-2) -curve and $((\rho \circ \pi)^*(\bar{l}) + l)l = 0$).

Let $B' \in |(\rho \circ \pi)^*(\bar{l}) + l|$ be a smooth element and A_1, \dots, A_9 be the disjoint (-2) -curves contained in $(\rho \circ \pi)^*(p)$. Let S be the minimal model of the double cover of W with branch locus $B' + \sum_1^9 A_i + l$. The surface Q is the image of the bicanonical map of S and $p_g(S) = 1$, $q(S) = 0$, $K_S^2 = 2$.

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